Tutorial 2

Q.1

Show that f and g are integrable on a bounded set A implies that the product fg is also integrable.

Prerequsite:

Recall the definition of integrability on bounded set.

Let A be bounded & $f: A \to \mathbb{R}$. We say f is integrable on A if $\exists$ rectangle $R \supseteq A$ s.t. $\bar{f}$ is integrable on R, where

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \in R \setminus A. \end{cases}$$

$$\int_A f \, dV := \int_R \bar{f}$$

Solution:

Since $fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$, it suffices to show that if f is integrable on A, then so is f^2.

WLOG, assume $f \not\equiv 0$. (The case when $f \equiv 0$ is trivial)

By def. of integrability, $\exists$ rectangle $R \supseteq A$ s.t. $\bar{f}$ is integrable on R.

In other words, $\forall \varepsilon > 0$, $\exists$ partition P of R s.t.

$\mathcal{U}(\bar{f}, P) - \mathcal{L}(\bar{f}, P) < \frac{\varepsilon}{2M}$, $M := \sup\limits_{x \in A} |f(x)|$

Note that $\overline{f^2} = (\bar{f})^2 = |\bar{f}|^2$, and $\begin{cases} \sup \overline{f^2} = (\sup |\bar{f}|)^2 \\ \inf \overline{f^2} = (\inf |\bar{f}|)^2 \end{cases}$

$$\therefore \mathcal{U}(\overline{f^2}, P) - \mathcal{L}(\overline{f^2}, P)$$

$$= \sum_{C \in P} \left(\sup_{x \in C} \overline{f^2} - \inf_{x \in C} \overline{f^2}\right) \mathrm{vol}(C)$$

$$= \sum_{C \in P} \left[\left(\sup_{x \in C} |\bar{f}|\right)^2 - \left(\inf_{x \in C} |\bar{f}|\right)^2\right] \mathrm{vol}(C)$$

$$= \sum_{C \in P} \left[\left(\sup_{x \in C} |\bar{f}|\right) - \left(\inf_{x \in C} |\bar{f}|\right)\right]\left[\left(\sup_{x \in C} |\bar{f}|\right) + \left(\inf_{x \in C} |\bar{f}|\right)\right] \mathrm{vol}(C)$$

$$\leq 2M \sum_{C \in P} \left(\sup_{x \in C} |\bar{f}| - \inf_{x \in C} |\bar{f}|\right) \mathrm{vol}(C)$$

$$\leq 2M \sum_{C \in P} \left(\sup_{x \in C} \bar{f} - \inf_{x \in C} \bar{f}\right) \mathrm{vol}(C)$$

$$= 2M\left(\mathcal{U}(\bar{f}, P) - \mathcal{L}(\bar{f}, P)\right)$$

$$< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$$

Q.2

Prove that any subset of measure zero set must have measure zero.

Solution:

A subset $A \subseteq \mathbb{R}^n$ is said to have measure zero if $\forall \varepsilon > 0, \exists$ a sequence of rectangles $\{R_i\}_{i=1}^{\infty}$ s.t.

(i) $A \subseteq \bigcup_{i=1}^{\infty} R_i$

(ii) $\sum_{i=1}^{\infty} \text{vol}(R_i) < \varepsilon$.

Let B be any subset of A. Given any $\varepsilon > 0$, we can just choose $\{R_i\}_{i=1}^{\infty}$ as mentioned above, so that we have

(i') $B \subseteq A \subseteq \bigcup_{i=1}^{\infty} R_i$

(ii') $\sum_{i=1}^{\infty} \text{vol}(R_i) < \varepsilon$

$\therefore$ B has measure zero.

Q.3

Give an example of a bounded measure zero subset whose boundary does not have measure zero.

Solution:

Let $A = \mathbb{Q} \cap [0,1] \subseteq \mathbb{R}^1$

<u>Claim 1</u>: A has measure zero.

<u>Proof</u>: Let $\{q_i\}_{i=1}^{\infty}$ be an enumeration of rational number in $[0,1]$. Given $\varepsilon > 0$, let $R_{i,\varepsilon} = [q_i - \frac{\varepsilon}{2^{i+2}}, q_i + \frac{\varepsilon}{2^{i+2}}]$.

Clearly, $A \subseteq \bigcup_{i=1}^{\infty} R_{i,\varepsilon}$.

Also, $\sum_{i=1}^{\infty} \text{vol}(R_{i,\varepsilon}) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2} < \varepsilon$. $\square$

<u>Claim 2</u>: $[0,1] \subseteq \partial A$, so ∂A does not have measure zero.

<u>Proof</u>: We first show that $[0,1] \subseteq \partial A$.

Let $x \in [0,1]$. Let (a,b) be an open interval containing x.

Note that $(a,b) \cap [0,1]$ is always an interval of positive length, so it must contain some rational number $\neq x$ & irrational number in $[0,1]$ by the density of $\mathbb{Q}$ & $\mathbb{Q}^c$, i.e. $(a,b) \cap A \neq \phi$ & $(a,b) \cap (\mathbb{R} \setminus A) \neq \phi$. $\therefore [0,1] \subseteq \partial A$.

Since $\text{vol}([0,1]) = 1$, it is impossible to cover ∂A with rectangles of total volume $< \varepsilon < 1$. $\therefore \partial A$ does not have measure zero. $\square$